Subgraph Isomorphism

Ullmann's Algorithm

- Given: Two graphs $G_A(V_A, E_A)$ and $G_B(V_B, E_B)$ and their adjacency matrices: A and B
- Idea: $n = |V_a|$, $m = |V_b|$, $n \times m$ ("permutation") matrix M with following form:
  - M contains only '0' and '1'
  - Exact one '1' in each row
  - Not more than one '1' in each column
- Permutate adjacency matrix B by multiplying it with M, and compare adjacency.

Subgraph Isomorphism

Ullmann's Algorithm

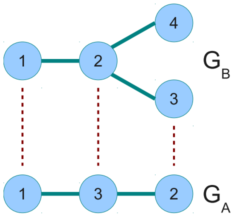
- $M \times B$: Move row j to row i $\forall M_{ij} = 1$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M = M^T} \times \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{B = B^T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- $(MB)^T$: Move column j to column i
- $M(MB)^T$: Move column j to column i and row j to row i

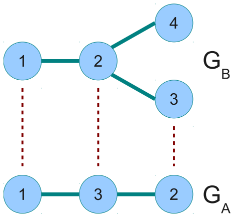
Subgraph Isomorphism

Ullmann's Algorithm



Subgraph Isomorphism

Ullmann's Algorithm



$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$B=B^T$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

M

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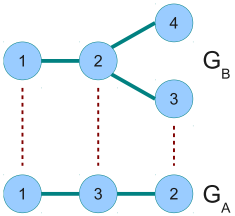


0	1	0	0
1	0	1	1
0	1	0	0
0	1	0	0

$B=B^T$

1	0	0	0
0	0	1	0
0	1	0	0

M

$$
\begin{aligned}
M(MB)^T &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right)^T \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = C
\end{aligned}
$$

Subgraph Isomorphism

Ullmann's Algorithm

also called "induced subgraph isomorphism"

Creating pairs of nodes by exchanging rows and columns (renaming).

Adjacency condition

Let $C = M(MB)^T$,
A is a (subgraph-) isomorphism iff

$$A_{ij} = 1 \Leftrightarrow C_{ij} = 1 \forall i, j$$

How do we get M?

Subgraph Monomorphism

Ullmann's Algorithm

- sometimes called "subgraph isomorphims problem" (sigh!)
- "most common" version

Creating pairs of nodes by exchanging rows and columns (renaming).

Adjacency condition

Let $C = M(MB)^T$,
A is a (subgraph-) monomorphism
iff

$$A_{ij}=1 \Rightarrow C_{ij}=1 \forall i, j$$

How do we get M?

Subgraph Isomorphism
Ullmann's Algorithm

- Build Startmatrix M^0 by setting all values to 1 (allow all permutations)
- Set values to 0 for all M^0_{ij} where $deg(B_j) < deg(A_i)$ (remove impossible permutations)

$$M^0_{ij} = \begin{cases} 1 & if \quad deg(B_j) \geq deg(A_i) \\ 0 & \text{otherwise} \end{cases} \quad , \forall i,j$$

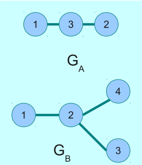
- Generate systematically permutation matrices M^d.

$$\underset{A}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \rightarrow \underset{M^0}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \leftarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Subgraph Isomorphism

Ullmann's Algorithm

1	1	1	1
1	1	1	1
0	1	0	0



Subgraph Isomorphism

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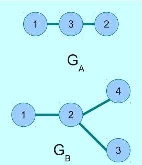
1	1	1	1
1	1	1	1
0	1	0	0

1	0	0	0
1	1	1	1
0	1	0	0



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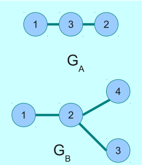
1	1	1	1
1	1	1	1
0	1	0	0

1	0	0	0
1	1	1	1
0	1	0	0

1	0	0	0
1	0	0	0
0	1	0	0



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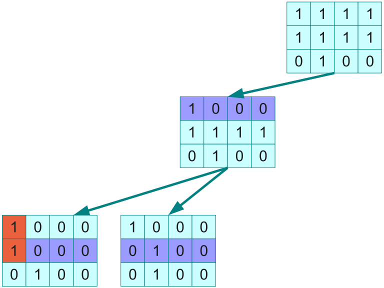

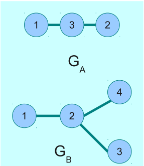

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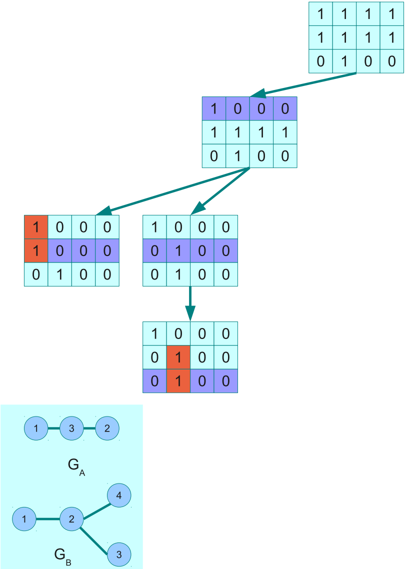

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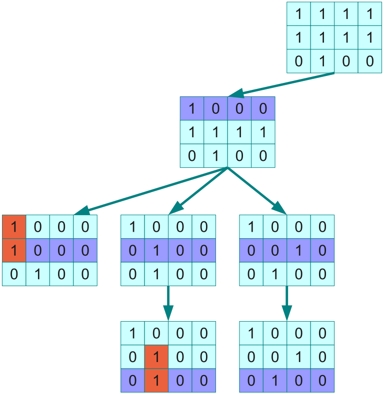

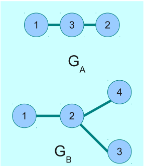

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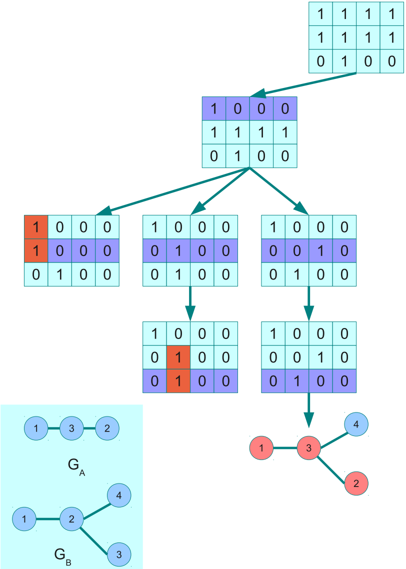

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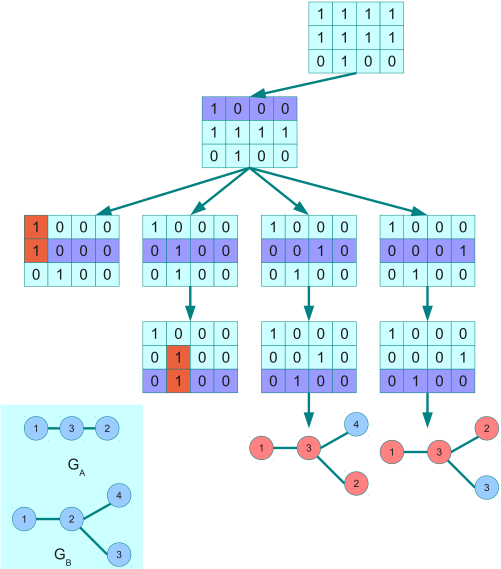

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